A Survey of Knot Concordance and L-space Knots

Thomas Sachen
tsachen@princeton.edu

November 7, 2022

1 Introduction

This paper is an exposition of some results in classical knot concordance with further attention given to modern techniques in knot theory. The aim is to gently build up the tools necessary to analyze a family of problems in knot theory concerning concordances of torus knots with L-space knots, a problem of some interest in current research. I'll give some examples of progress made by several authors along these lines, and I'll conclude by presenting some of my own results and discussing further directions for research.

I'll begin by giving a background in torus knots and knot concordance, concluding with Litherland's formula and the proof that torus knots are linearly independent in the concordance group. Then we'll begin a deeper discussion of various concordance invariants of interest before going into an exposition of L-space knots and some stepping-stones on the way to generalizations of Litherland's work.

2 Knot Concordance

An oriented knot K is said to be *smoothly slice* if it is the boundary of a disk D^2 smoothly embedded in the 4-ball D^4, or in other words if there is a pair (D^4, D^2) with $\delta(D^4, D^2) = K$. There is a related, slightly weaker notion, of a *topologically slice* knot, which bounds an embedded disk that need only be topologically locally flat. Unless otherwise noted, we'll be discussing the smooth case exclusively.

Two oriented knots K and J are said to be *smoothly concordant* if $K\# -J$ is smoothly slice, where $\#$ denotes the connected sum of knots and $-K$ denotes the knot obtained by reversing the orientation of K. One can use basic properties of the connected sum to deduce the following proposition.

Proposition 2.1. *Concordance is an equivalence relation on the set of oriented knots.*

In particular, this entails that $K\# -K$ is slice for any oriented knot K.

Equivalently, K and J are concordant if they cobound a smoothly embedded cylinder in D^4. More precisely, K and J are concordant if and only if there is a pair (D^4, C) with $\delta(D^4, C) = K \sqcup J$, where C is a smooth 2-manifold diffeomorphic to $S^1 \times [-1, 1]$ with $S^1 \times \{-1\} = K$ and $S^1 \times \{1\} = J$. This characterization of knot concordance is useful in proving that the set of concordance classes of knots forms an abelian group $\mathcal{C}$, where the group law is connected sum and the identity element is the (concordance class of the) unknot. The critical reader might question whether the connected sum operation is well-defined on concordance classes – after all, the knot diagrams for two concordant knots might look entirely different. One can convince himself of this somewhat non-obvious fact by again arguing using this characterization of concordance in terms of cobordisms.

Proposition 2.2. *The set of concordance classes of oriented knots along with the connected sum operation form an abelian group.*

The group so formed is called the (smooth) *concordance group* $\mathcal{C}$, and it has been an object of considerable importance in knot theory since its introduction by Fox and Milnor in 1966 [9]. Once again, there is a related notion in the topologically locally flat category.

A few remarks are in order. Elements in the concordance group are not oriented knots themselves, but rather equivalence classes of knots with respect to the concordance relation.

Therefore it isn't correct, for instance, to say that the unknot is the identity element in $\mathcal{C}$; rather, the concordance class of the unknot is the identity element in $\mathcal{C}$. In other words, any slice knot is in the identity class in $\mathcal{C}$.

One might wonder whether *all* knots are slice. This is not true; in fact the trefoil knot is not slice, and it is of infinite order in $\mathcal{C}$. This will be shown later with the help of some concordance invariants. Furthermore $\mathcal{C}$ is quite "large" – in 1969 Levine found a surjective homomorphism $\phi : \mathcal{C} \twoheadrightarrow \mathbb{Z}^\infty \oplus \mathbb{Z}_2^\infty \oplus \mathbb{Z}_4^\infty$, with Casson and Gordon showing in 1975 that this homomorphism is not an isomorphism [9]. There is much work on the concordance group and its properties, but we will only discuss those relevant to this paper. For now, we turn our attention to a useful concordance invariants.

2.1 Seifert Matrices and Signature

A *Seifert surface* for a knot K is pair $F = (S^3, M)$ with $\delta F = K$, where M is a compact, connected, oriented 2-manifold, where the orientation of K is induced by the orientation of M. Note that the Seifert surface of a particular knot need not be unique up to diffeomorphism, and furthermore that every knot has a Seifert surface which can be obtained concretely using a simple technique called Seifert's algorithm.

Seifert surfaces are bicollarable, meaning that to a Seifert Surface F we can associate a *bicollar* $F \times [-1, 1]$ in S^3. If x is a 1-cycle in $H_1(F)$, let x^+ denote the 1-cycle carried by $x \times \{1\}$ in the bicollar, and similarly let x^- denote the 1-cycle carried by $x \times \{-1\}$. We can define a bilinear form

$$\alpha : H_1(F) \times H_1(F) \to \mathbb{Z}, \text{ with } \alpha(x, y) = \mathrm{lk}(x, y^+),$$

where lk denotes the linking number of x and y. Proof that α is defined and bilinear isn't relevant here, but can be found in [6]. Note also that $\mathrm{lk}(x, y^+) = \mathrm{lk}(x^-, y)$.

The bilinear form α is called the *Seifert form* of F. Furthermore, choosing a basis $\{e_i\}$ of $H_1(F)$ viewed as a $\mathbb{Z}-$module, we can associate to F its *Seifert matrix* V, a square matrix with entries $v_{ij} = \alpha(e_i, e_j)$. The Seifert matrix of a knot is not itself a knot invariant, as it may differ for different choices of Seifert surfaces of a given knot. We'll often work with the more amenable matrix $V + V^T$, as it's symmetric and non-singular for knots (this is an easy exercise).

With the Seifert matrix in hand, we can define an important concordance invariant. The *signature* of a knot K, denoted $\sigma(K)$, is defined as $\sigma(V + V^T)$, To see that signature is well-defined (that is, independent of our choice of Seifert surface) and is indeed a concordance invariant, refer to [12]. For our purposes, we need only note the following basic properties, for oriented knots K and J.

Proposition 2.3. *$\sigma(K\#J) = \sigma(K) + \sigma(J)$.*

Proposition 2.4. *If K is slice, then $\sigma(K) = 0$.*

Example 2.5. Signature is enough to show that the trefoil isn't slice. The right-handed trefoil $T(2,3)$ has Seifert matrix

$$V = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix},$$

which can easily be computed using Seifert's algorithm to obtain a Seifert surface for $T(2,3)$. V has signature -1, and in particular $T(2,3)$ is not slice.

A useful generalization of knot signature is the Tristram-Levine signature, developed in the late 1960s. The Tristram-Levine signature for a knot K is a function $\sigma_K : S^1 \to \mathbb{Z}$ which associates to each point ω on the complex unit circle a signature function $\sigma_K(\omega)$ defined by

$$\sigma_K(\omega) = \sigma((1-\omega)V + (1-\overline{\omega})V^T),$$

where V is any Seifert matrix for K and σ is interpreted in the usual sense as the number of positive eigenvalues minus the number of negative eigenvalues of a matrix. For a slice knot K, $\sigma_K(\omega)$ vanishes almost everywhere on S^1, possibly taking on non-zero values at roots of the *Alexander polynomial* $\Delta_K(t) \doteq \det(tV - V^T)$,[1] another knot invariant. Furthermore for general knots, $\sigma_K(\omega)$ has discontinuities only at roots of Δ_K.

A related notion is the the *determinant* of a knot K, denoted $\det(K)$ and defined as $\det(V + V^T)$. Note that the determinant of a knot *is not* a concordance invariant; rather, it is only a knot invariant. We note the following properties, which will be useful later.

Proposition 2.6. $\det(K\#J) = \det(K)\det(J)$.

Proposition 2.7. *The determinant of a slice knot is a square integer.*

There is of course much more to be said about properties of the determinant and the Tristram-Levine signature, but for now we've built up enough background to discuss Litherland's theorem and the behavior of torus knots in the concordance group.

3 Torus Knots and Litherland's Theorem

A *torus knot* is a knot which is contained in an unknotted torus. In fact, any torus knot K can be uniqely specified up to ambient isotopy by a pair of coprime integers (p, q), with p the number of times K winds around the torus longitudinally, and q the number of times K winds around the torus meridianally. An easy and enlightening exercise is to show why K fails to be a knot when p and q are not coprime.

The point of this article is to work towards a generalization of a key theorem of Litherland: that torus knots are linearly independent in the concordance group. We first summarize and sketch the results of Litherland, adapted from his 1979 paper [7].

[1] The use of $\doteq$ signifies that the Alexander polynomial is defined only up to multiples of t and t^{-1}: see [6] for more details and proof of these properties.

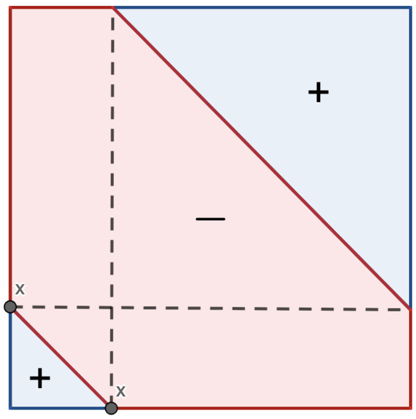

Figure 1:

Proposition 3.1 (Litherland's formula)**.** *If K is a (p,q) torus knot and $\zeta = e^{2\pi i x}$, x rational, $0 < x < 1$, then $\sigma_\zeta(K) = \sigma_{\zeta^+}(K) - \sigma_{\zeta^-}(K)$, where*

(i) $\sigma_{\zeta^+}(K) =$ *number of pairs (i, j) of integers, $0 < i < p, 0 < j < q$, such that* $x - 1 < \frac{i}{p} + \frac{j}{q} < x \mod 2$

(ii) $\sigma_{\zeta^-}(K) =$ *number of pairs such that* $x < \frac{i}{p} + \frac{j}{q} < x + 1 \mod 2$

I'll give a brief overview of structure of the proof. More details can be found in [7], though the workings of the proof aren't relevant here. To give some geometric meaning to this rather messy looking formula, we may equivalently interpret $\sigma_\zeta(K)$ as the signed sum of the number of integer lattice points inside the regions in the interior of the unit square indicated in figure 1.

To this quantity we may associate a jump function $f_{(p,q)}$ as a function of x, and from the diagram it should be apparent that $f_{(p,q)}(x)$ is the number of integer lattice points on the lower diagonal line minus the number of integer lattice points on the upper diagonal line of the above figure. With a bit more work, it's actually possible to compute $f_{(p,q)}$ explicitly, as

$$f_{(p,q)} = \begin{cases} h_{(p,q)}(pqx) & \text{if } pqx \in \mathbb{Z} \text{ and } px, qx \notin \mathbb{Z} \\ 0 & \text{otherwise,} \end{cases}$$

where $h_{(p,q)} = (-1)^{[a/q]+[b/p]+[n/pq]}$ for any integer $n = ap+bq$. Even more, a few combinatorial tricks allow us to show that the family of functions $\{f_{(p,q)}\}$ are linearly independent, in the sense that no linear combination of these functions vanishes identically.

Corollary 3.2. *Torus knots are linearly independent in the smooth concordance group.*

Proof. Since the jump function $f_{(p,q)}$ is a concordance invariant of a $T(p,q)$ torus knot, we may conclude that the torus knots are linearly independent in $\mathcal{C}$. □

Litherland's work provides the motivation for the culminating issue. While Litherland was able to show the linear independence of torus knots in $\mathcal{C}$, no analogous result has since been found for the class of *algebraic knots*, let alone for the larger class of L-space knots, though some progress has been made.

4 L-spaces and L-space knots

4.1 Heegaard Floer homology

Heegaard Floer homology is a powerful tool for analyzing 3-manifolds, developed by Oszvath and Szabo in the early 2000s. Among many other things, two early applications of Heegaard Floer homology were to bound the 4-ball genus of knots[2] and to detect whether knots are fibered[3] [5]. In order to give the definition of a basic form of Heegaard Floer homology, we

[2]The 4-ball genus of a knot K is the minimal genus among all orientable 2-manifolds properly embedded in D^4 with boundary K. For example, slice knots have 4-ball genus 0.

[3]A knot K is *fibered* if there is a continuous function $f : S^1 \to D^4$ associating to each point x on the complex unit circle a Seifert surface F_x of K that intersects F_y at precisely K whenever $x \neq y$.

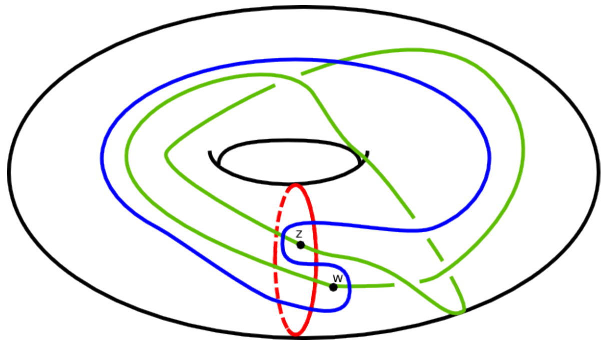

Figure 2: A Heegaard diagram for the trefoil $T(2,3)$. Here the red circle is the set α and the blue circle is the set β.*

∗ Figure adapted from https://maths.dur.ac.uk/users/mark.a.powell/HFHseminar.html

must first introduce some terminology.

A *handlebody of genus* g is a compact, orientable[4] 3-manifold obtained by attaching g handles to D^3. More precisely, we may select two disjoint 2-disks D_1 and D_2 on the surface of the 3-ball D^3 and glue in a copy of $S^1 \times [-1,1]$ such that $S^1 \times \{-1\} = \delta D_1$ and $S^1 \times \{1\} = \delta D_2$. Surprisingly, every 3-manifold admits a *Heegard splitting,* or decomposition into two handlebodies. Any two handlebodies of the same genus are of course homeomorphic, so we can identify their boundaries via a (possibly quite complicated) homeomorphism to obtain a new 3-manifold.

Further, any Heegaard splitting is completely determined by a *pointed Heegaard diagram* (Figure 2) specifying a set $\boldsymbol{\alpha}$ (resp. $\boldsymbol{\beta}$) of g many pairwise-disjoint circles $\alpha_1, \ldots, \alpha_g$ (resp. $\beta_1, \ldots, \beta_g$) and a basepoint z contained in a closed, oriented surface Σ of genus g. We also require that each circle spans a $g-$dimensional subspace of Σ and that α_i and β_j intersect transversally (if at all) for all $i, j \leq g$. To recover a 3-manifold from a Heegaard diagram, we can thicken Σ, attach thickened disks along the α_i and β_i, and glue in copies of D^3 for each remaining boundary component, noticing that each component is homeomorphic to S^2.

In its simplest form, Heegaard Floer homology associates to each closed 3-manifold Y a chain complex $\widehat{CF}(Y)$ whose chain homotopy type is an invariant of the manifold. In

[4]Handlebodies are not always assumed to be orientable, but it is enough for our purposes.

particular, the generators of $\widehat{CF}(Y)$ are the points in $\mathbb{T}_a \cap \mathbb{T}_b$, where $\mathbb{T}_a = \alpha_1 \times \cdots \times \alpha_g$ and $\mathbb{T}_b = \beta_1 \times \cdots \times \beta_g$. Then we can define a differential operator

$$\hat{\partial}\boldsymbol{x} = \sum_{\mathbb{T}_a \cap \mathbb{T}_b} \sum_{\{\phi \in \pi_2(\boldsymbol{x},\boldsymbol{y}) | \mu(\phi)=1, n_w(\phi)=0\}} \# \left(\frac{\mathcal{M}(\phi)}{\mathbb{R}} \right)$$

and let $\widehat{HF}(Y)$ be the homology thus obtained. The precise meaning or motivation for this definition is beyond the scope of this writing (very thorough treatments can be found in [11] or [5]), but I included it to show that $\widehat{HF}$ is defined very concretely and is theoretically computable. Furthermore, a knot K in a 3-manifold Y induces a filtration on $\widehat{CF}(Y)$, giving rise to the *Heegaard Floer knot complex* $CFK^\infty(K)$, which will be relevant later.

Indeed, we have the following key inequality [5]:

$$\dim \widehat{HF}(Y) \geq |H_1(Y;\mathbb{Z})|,$$

where the right-hand side is just the usual integral homology of Y and $\widehat{HF}(Y)$ is regarded as a graded vector space over some field $\mathbb{F}$. Of particular interest are those 3-manifolds Y for which the dimension of $\widehat{HF}(Y)$ is minimal, in the sense that $\dim \widehat{HF}(Y) = |H_1(Y;\mathbb{Z})|$. Such 3-manifolds are called *L-spaces*.

4.2 L-space knots

Knots which admit positive Dehn surgery to L-spaces are *L-space knots.* Recall that Dehn surgery is a method for constructing new 3-manifolds out of old ones. Given a 3-manifold M and a link L, we can subtract a tubular neighborhood of L from M to obtain the *link complement* of L in M, diffeomorphic to $M \setminus L$. Each boundary component of the resulting link complement is homeomorphic to the torus $\mathbb{T}^2$, so we can glue a solid torus into each component of the link complement via a homeomorphic identification of the boundaries to obtain a new, surgered 3-manifold.

L-spaces are natural generalizations of *lens spaces,* 3-manifolds obtained by gluing two solid tori together via a homeomorphism of their boundaries; all lens spaces are L-spaces [11]. And since all torus knots admit lens space surgeries [9], a natural follow-up to Litherland's theorem is whether all L-space knots are linearly independent in the concordance group. Answering this question will resolve the decades-old problem of whether *algebraic knots* are linearly independent in $\mathcal{C}$, as all algebraic knots are themselves L-space knots.

5 Concordances of torus knots with L-space knots

A promising jumping-off point, then, is to answer the question of whether any linear combination of torus knots is concordant to an L-space knot. A recent result from Livingston answered the question affirmatively for connected sums of *positive torus knots.* That is, torus knots $T(p, q)$ for which $p, q \geq 2$

Theorem 5.1 (Livingston, 2018)**.** .

Let $\{(p_i, q_i)\}_{i=1,\ldots,n}$ *be a set of pairs of relatively prime integers with* $2 \leq p_i \leq q_i$ *for all* i *and with* $n > 1$*. Then* $\#_i T(p_i, q_i)$ *is not concordant to an L-space knot.*

In 2019, Allen gave another partial result toward the more general case, this time concerning differences of multiples of torus knots.

Theorem 5.2 (Allen, 2019)**.** *If the connected sum of distinct positive torus knots* $mT(p, q)\#nT(r, s)$ *is concordant to an L-space knot, then either* $m = 0$ *and* $n = 1$ *or* $m = 1$ *and* $n = 0$*.*

To prove these results, Livingston and Allen relied on a combination of classical and modern techniques. As noted by Livingston, usual knot Floer homology cannot distinguish, for example, $K = T(2, 3)\#T(2, 3)$ from $J = T(2, 5)$ up to concordance, even though J is an L-space knot and K was shown independently to not be concordant to an L-space knot. Both

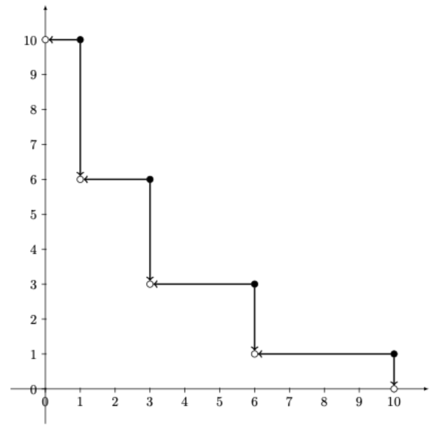

Figure 3: $CFK^\infty(T(5,6))$ is a staircase of the form $[1,4,2,3,3,2,4,1]^*$
$*$ Figure adapted from [2]

$CFK^\infty(K)$ and $CFK^\infty(J)$ are staircases. Furthermore the height and width of each step is the same, as these features are determined by the gaps in the exponents of the Alexander polynomial of K (or J). However, $CFK^\infty(J)$ is formed by adding an acyclic summand to $CFK^\infty(K)$, and so CFK^∞ and invariants arising therefrom are not quite enough.[5]

The most general form of this conjecture was given by Allen [2].

Conjecture 5.3 (Allen, 2019). *If a connected sum of (possibly several) torus knots is concordant to an L–space knot, then it is concordant to a positive torus knot.*

A reasonable jumping-off point to work toward this conjecture is to analyze *2-bridge knots* – knots of bridge number 2. A fact not mentioned earlier is that the determinant of a knot K is equal to $|H_1(\tilde{X})|$, where $\tilde{X}$ denotes the two-fold branched cover of K[12]. Since the two-fold branched cover of a two-bridge knot is a lens space, we can analyze the two-fold branched cover of linear combinations of two-bridge knots and analyze the behavior of the determinant, exploiting the fact that determinant is a concordance invariant.

[5]It should be noted that this isn't technically true – at least, it's a bit misleading. Zemke pointed out that the involutive Floer homology introduced by Hendricks, Manolescu [1507.00383] and Zemke [1705.01117] resolve this issue. The knot complexes for K and J thus obtained are actually distinguishible – the complex for K is a "box" and the complex for J is a staircase. Still, I included this section as it's the justification given by Livingston for his use of classical invariants in lieu of more modern techniques.

Recent work by Aceto-Celoria-Park is quite useful in this regard. Note that a cobordism on the level of two-fold branched covers of knots corresponds to a concordance on the level of knots. Letting $\mathcal{L}$ denote the subgroup of $\Theta^3_{\mathbb{Q}}$, the 3-dimensional $\mathbb{Q}$-homology cobordism group, generated by lens spaces, we have the following results.

Theorem 5.4 (Aceto-Celoria-Park, 2018)**.** *Any class in $\mathcal{L}$ contains a connected sum of lens spaces L such that if Y is $\mathbb{Q}$-homology cobordant to L, then there is an injection*

$$H_1(L;\mathbb{Z}) \hookrightarrow H_1(Y,\mathbb{Z}).$$

Moreover, as a connected sum of lens spaces L is uniquely determined up to orientation preserving diffeomorphism.

Corollary 5.5 (Aceto-Celoria-Park, 2018)**.** *Any smooth concordance class in the subgroup generated by 2-bridge knots is represented by a connected sum of 2-bridge knots K such that if J is concordant to K, then* $\det(J)$ *divides* $\det(K)$*. Moreover, as a connected sum of 2-bridge knots K is uniquely determined up to isotopy.*

In other words, each class in the subgroup of the smooth concordance group generated by two-bridge knots has a "minimal element" whose determinant divides the determinants of all other knots in its concordance class. This result gives enough motivation to continue the work of Allen and Livingston, in the more restricted setting of two-bridge knots. Hence, the following is conjectured.

Conjecture 5.6. *No connected sum of 2-bridge knots and their mirrors is concordant to an L-space knot.*

Let K_2^+ (resp. K_2^-) denote a connected sum of positive (resp. negative) $T(2,q)$ torus knots. Similarly, we shall use K^+ (respectively K^-) to denote an arbitrary connected sum of positive (respectively negative) $T(p,q)$ torus knots. Noting that all $T(2,q)$ torus knots are 2-bridge knots, a partial result on the way to this more general conjecture is the following.

Theorem 5.7. *Let $K = K_2^+ \# K_2^-$. Then K is concordant to an L-space knot J if and only if $K = T(2,q)$ for some $q \in \mathbb{Z}_{\geq 1}$.*

Before giving the proof of Theorem 5.7, we need a lemma.

Lemma 5.8. *If $K = K^+ \# K^-$ is concordant to an L-space knot J, then* $\det(J) = \frac{\det(K^+)}{\det(K^-)}$.

Proof. Proposition 3.3 of [2] allows us to compute $\det(J)$ directly. Since K is concordant to an L-space knot, if we write K as $K = T_{p_1,q_1} \# T_{p_2,q_2} \# \cdots \# T_{p_m,q_m} \# -T_{p_1',q_1'} \# -T_{p_2',q_2'} \# \cdots \# -T_{p_n',q_n'}$ where $m, n \geq 1$, we have the following formula:

$$\frac{\prod_{i=1}^m \Delta_{T_{p_i,q_i}}(t)}{\prod_{i=1}^n \Delta_{T_{p_i',q_i'}}(t)} = \Delta_J(t).$$

And since $\det(J) = |\Delta_J(-1)|$, we can simplify the above to obtain

$$\begin{aligned}\det(J) = \left| \frac{\prod_{i=1}^m \Delta_{T_{p_i,q_i}}(-1)}{\prod_{i=1}^n \Delta_{T_{p_i',q_i'}}(-1)} \right| &= \frac{\prod_{i=1}^m \left|\Delta_{T_{p_i,q_i}}(-1)\right|}{\prod_{i=1}^n \left|\Delta_{T_{p_i',q_i'}}(-1)\right|} \\ &= \frac{\det(\#_i T(p_i,q_i))}{\det(\#_i T(p_i',q_i'))} = \frac{\det(K^+)}{\det(K^-)}\end{aligned}$$

□

Before undertaking the proof of Theorem 4.7 , we first comment on the nature of a *reduced* connected sum of lens spaces. The unique minimal element in a class of $\mathcal{L}$ referenced in Theorem 5.4 is the unique (up to orientation-preserving diffeomorphism) *reduced* connected sum of lens spaces defined precisely by Aceto-Celoria-Park [1]. In particular, an $L(1,m)$ lens space is reduced if and only $m \neq 4$.

Proof. The two-fold branched cover of $T(2,q)$ is $L(q,q-1)$, so the two-fold branched cover $\tilde{K}$ of K is a connected sum of $L(q_n,q_n-1)$ lens spaces. Since K is a connected sum of $T(2,q)$ torus knots, $q_n \neq 4$ for any n and so each summand in $\tilde{K}$ can be easily seen to be reduced.

Therefore $\tilde{K}$ is the unique reduced representative of lens spaces in $[J]$, the cobordism class of J in $\mathcal{L}$. Denoting $A = \det(K_2^+)$ and $B = \det(K_2^-)$, we computed earlier that $\det(K) = AB$ and $\det(J) = A/B$. Then Corollary 5.5 ensures that $AB|A/B$, so in particular $B = 1$ and K_2^- is the unknot. Hence K is a connected sum of only positive torus knots, so the result follows immediately from Theorem 5.1. □

Having established the desired result for connected sums of $T(2,q)$ torus knots and their mirrors, we move to the more general case of connected sums of 2-bridge knots and their mirrors, with the hope of eventually proving Conjecture 5.6. The hope is to make more headway on the problem of showing the linear independence of L-space knots in $\mathcal{C}$ by showing the linear independence of several subclasses of L-space knots.

References

[1] Paolo Aceto, Daniele Celoria, and JungHwan Park. "Rational cobordisms and integral homology". *Compos. Math* 156.9 (2020), pp. 1825-1845. ISSN:0010-437X.

[2] Samantha Allen. "Concordances from differences of torus knots to L-space knots". *Proc. Amer. Math. Soc.* 148.4 (2020), pp1815-1827. ISSN:0002-9939.

[3] Anthony Conway. (2021). The levine-tristram signature: A survey. 2019-20 MATRIX Annals, 31–56. https://doi.org/10.1007/978-3-030-62497-2_2

[4] Kristen Hendricks and Ciprian Manolescu. "Involutive Heegaard Floer homology." Duke Math. J. 166 (7) 1211 - 1299, 15 May 2017. https://doi.org/10.1215/00127094-3793141

[5] Jennifer Hom. A survey on Heegaard Floer homology and concordance, J. Knot Theory Ramifications 26 (2017), no. 2, 1740015, 24. MR 3604497

[6] Lickorish, W. B. R. (1997). An introduction to knot theory. Springer.

[7] R.A. Litherland. (1979) Signatures of iterated torus knots. In: Fenn R. (eds) Topology of Low-Dimensional Manifolds. Lecture Notes in Mathematics, vol 722. Springer, Berlin, Heidelberg. https://doi.org/10.1007/BFb0063191

[8] Charles Livingston. "Concordances from connected sums of torus knots to L-space knots". *New York J. Math.* 24 (2018), pp. 233-239.

[9] Menasco, W. W., Thistlethwaite, M., & Livingston, C. (2005). A Survey of Classical Knot Concordance. In Handbook of Knot Theory. essay, Elsevier Science.

[10] Louise Moser. Elementary surgery along a torus knot. Pacific J. Math., 38:737–745, 1971.

[11] Peter Ozsváth and Zoltán Szabó. "Knot Floer homology and the four-ball genus". *Geom. Topol.* 7 (2003), pp. 615-639. ISSN: 1465-3060

[12] Rolfsen, D., *Knots and links*, Mathematics Lecture Series, No. 7. Publish or Perish, Inc., Berkeley, Calif., 1976.

[13] Jacob Andrew Rasmussen. "Floer homology and knot complements". Thesis (Ph.D.)-Harvard University ProQuest LLC, Ann Arbor, MI, 2003, p. 126. ISBN:978-0496-39374-9.

[14] Ian Zemke. (2019). Connected sums and involutive knot Floer homology. Proceedings of the London Mathematical Society, 119(1), 214–265. https://doi.org/10.1112/plms.12227