p-adic Hodge Theory and Derived Hodge-to-de Rham Spectral Sequences

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Abstract

This paper gives an overview of Alexander Beilinson's novel construction of the p-adic period map and elegant proof of Fontaine's C_{dR} conjecture, comparing the de Rham and p-adic étale cohomology theories of varieties over algebraic closures of p-adic fields. This overview is given in parallel with the classical de Rham comparison theorem to demonstrate the elegance of Beilinson's approach. I also discuss various methods, focusing on spectral sequences, devised by Illusie, Bhatt, and others to compute algebraic de Rham cohomology and derived de Rham cohomology of certain classes of schemes and p-adic varieties.

Chapter 0

Introduction

When I first encountered de Rham's theorem, I was fascinated by the possibility of using topological data to understand analytic properties of mathematical objects. This connection first seemed so deep as to be impossible, but once the reasoning was laid in front of me it felt like an inevitability. I wanted to push these ideas further: to understand which properties of manifolds or algebraic varieties were due to some deep underlying phenomena rather than peculiarities of their settings. For example, Hodge symmetry holding for compact Kähler manifolds is usually explained in complex analysis courses as a byproduct of complex conjugation on the Dolbeault cohomology groups, but this doesn't touch on the deeper question of why it holds for abelian varieties in general.

I had an even stronger reaction when I encountered Fontaine's C_{dR} conjecture, comparing the de Rham and p-adic étale cohomology theories of p-adic varieties, which are constructed in such constrasting fashions – one is purely geometric and one is algebraic and arithmetic. Though Falting proved this conjecture in 1988 [6], the techniques used were quite complex and seemed to be completely out of touch with the classical de Rham theory. Alexander Beilinson breathed new life into the problem in 2012, with a paper proving not only the p-adic comparison isomorphism, but also a very natural analog of the classical Poincaré lemma, via an ingenious construction that feels as if it's scratching at the deep pattern underpinning these comparison theorems in general. I will give an exposition of Beilinson's construction of the comparison isomorphism and the p-adic Poincaré lemma, presenting it in parallel to the classical story. I will also discuss more recent work of Bhatt computing algebraic and derived de Rham cohomologies.

Chapter 1

Hodge theory and comparison isomorphisms

1.1 Classical Hodge Theory

One of the primary aims of classical Hodge theory is to understand the extent to which the cohomology of a complex manifold is controlled by its geometry. The principle objects of interest are *Kähler manifolds* – a complex manifold X is Kähler if it is equipped with an everywhere non-degenerate real closed 2-form ω, called the Kähler form. The mere existence of such a Kähler form already forces a surprising amount of structure on the manifold itself. Relatively straightforward analysis of these Kähler manifolds already yields several basic properties guaranteed solely by the existence of a Kähler form ω. Among others, we have that

(i) ω is a real closed differential form of type $(1,1)$ with respect to the complex structure on X.

(ii) Along with its complex structure, X also admits compatible Riemannian and symplectic structures.

(iii) If X is compact and of dimension $2n$, then the closed form ω^k is not exact for any integer $1 \leq k \leq n$.

(iv) If X is compact and E is a holomorphic bundle over X, then $\mathbb{P}(E)$, the projective bundle associated to E, is also compact Kähler.

(v) If Y is a complex compact submanifold of X, then $\widetilde{X}_Y$, the blowup of X along Y, is also Kähler, and furthermore it is compact if X is.

Let X be a compact Kähler manifold. Let $\delta, \bar{\delta}$ denote the Dolbeault operators $\delta : \Omega^{p,q} \to \Omega^{p+1,q}$ and $\bar{\delta} : \Omega^{p,q} \to \Omega^{p,q+1}$, where $\Omega^{p,q}$ is the vector bundle of complex differential (p,q)-forms on X, and let $d = \delta + \bar{\delta}$ be the exterior derivative (note that this relation holds in particular because X is a complex manifold). For a complex differential operator f, recall that the *f-Laplacian* Δ_f is the operator $\Delta_f = ff^* + f^*f$, where $f^\star$ denotes the formal adjoint operator of f. We say a differential form α on X is *Δ_f-harmonic* if it is annihilated by the f-Laplacian, *i.e.* if $\Delta_f \alpha = 0$.

If $A^k(X) = \bigoplus_{p+q=k} \Omega^{(p,q)}$ is the vector space of differential forms of total degree $p+q=k$ on X, the $\bar{\delta}$-Laplacian is compatible with this decomposition of $A^k(X)$ into types, and in particular if a form $\alpha \in A^k(X)$ is $\Delta_{\bar{\delta}}$ harmonic, each of its components $\alpha^{p,q}$ is $\Delta_{\bar{\delta}}$ harmonic as well. A key theorem is the following [21].

Theorem 1. *If X is Kähler, then $\Delta_d = 2\Delta_\delta = 2\Delta_{\bar{\delta}}$.*

Hence the Δ_d Laplacian respects this decomposition of forms into types as well. Thus the space $\mathscr{H}^k(X)$ of harmonic k-forms on X is the direct sum of the spaces $\mathscr{H}^{p,q}$:

$$\mathscr{H}^k(X) = \bigoplus_{p+q=k} \mathscr{H}^{p,q}(X). \tag{1.1}$$

Recall the key fact in the theory of de Rham cohomology that the de Rham cohomology classes of X (thus Betti cohomology classes, by de Rham's theorem) each have precisely one harmonic representative, yielding an isomorphism from the space of complex-valued harmonic k-forms on X to the Betti cohomology group $H^k(X,\mathbb{C})$. This combined with (1.1), yields the *Hodge decomposition* of the Betti cohomology of X

$$H^k(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}, \tag{1.2}$$

where $H^{p,q}$ denotes the set of classes of differential forms representable by a closed form of type (p,q). Note that we can replace the summands in the right hand side of 1.2 with any term appearing in the chain of isomorphisms

$$H^{p,q}(X) \xrightarrow{\sim} \mathscr{H}^{p,q}(X) \xrightarrow{\sim} H^q(X, \Omega_X^p),$$

where the last term is the qth cohomology of Ω_X^p viewed as a sheaf over X.

The key idea enabling this entire discussion, which will be of immense importance when we move to the p-adic case, is the Poincaré lemma, a deceptively simple statement that lends a tremendous amount of structure to differential forms on complex manifolds. In this setting, the simplest expression of this lemma is as follows.

Lemma 1 (Poincaré, version 1)**.** *Let α be a closed differential form of strictly positive degree on a differentiable manifold. Then, locally, there exists a differential form β such that $\alpha = d\beta$.*

While this version yields the helpful mantra "closed forms are locally exact", it is not quite as useful in practice as the following equivalent version, in terms of augmentations of chain complexes. Let $\mathbb{C}_X$ denote the constant sheaf of stalk $\mathbb{C}$ over X, and let $\Omega_X^\bullet$ be the de Rham complex of holomorphic differential forms on X.

Lemma 2 (Poincaré, version 2)**.** *The natural augmentation map*

$$\mathbb{C}_{X^\bullet} \to (\mathcal{O}_X \to \Omega_X^1 \to \Omega_X^2 \to \dots) \tag{1.3}$$

is a quasi-isomorphism,

where here an augmentation is understood to be a morphism in the category of simplicial objects, and $\mathbb{C}_{X^\bullet}$ is the constant simplicial object associated to $\mathbb{C}_X$ [20]. The equivalence of these statements is evident: version 2 implies that the de Rham cohomology of X is locally trivial (in different terms, the complex of sheaves $0 \to \mathbb{C} \to \mathcal{O}_X \to \Omega_X^1 \to \Omega_X^2 \to \dots$ is exact) so that, in the local picture, all holomorphic closed forms are exact. In the language of derived geometry, there is an isomorphism of right derived functors, where the augmentation above is viewed as a resolution of the constant sheaf $\mathbb{C}_X$:

$$R\Gamma(X, \mathbb{C}) \xrightarrow{\sim} R\Gamma(X, \Omega_X^\bullet). \tag{1.4}$$

The quasi-isomorphism appearing in the Poincaré lemma is precisely what induces the famous de Rham theorem comparing Betti and de Rham cohomology.

Theorem 2 (de Rham comparison)**.** *Let X be a complex manifold. Then there is a natural isomorphism*

$$H^*(X, \mathbb{Q}) \otimes \mathbb{C} \xrightarrow{\sim} H^*(X, \Omega_X^\bullet) \tag{1.5}$$

between Betti and de Rham cohomology.

This isomorphism, sometimes called a *period isomorphism*, can actually be described quite explicitly. Viewing the right hand side as $\mathrm{Hom}(H_*(\Omega_X^\bullet), \mathbb{C})$, this isomorphism associates a differential form $\omega \in H^*(X, \mathbb{Q})$ to the map $\gamma \mapsto \int_\gamma \omega$, where γ is a singular chain in X. The numbers thus obtained by integration are known as *periods.*.

Indeed, even "version 2" of the Poincaré lemma fails in general when we consider the case of schemes over $\mathbb{C}$, rather than manifolds or smooth varieties. Let X be a smooth scheme of finite type over $\mathbb{C}$, and denote by X^{an} its analytification. Equivalently, the corresponding map of right derived functors

$$R\Gamma(X, \mathbb{C}) \to R\Gamma(X^{\mathrm{an}}, \mathbb{C}) \tag{1.6}$$

fails to be an isomorphism.

A deep theorem of Grothendieck [10] says that the natural map

$$R\Gamma(X, \Omega^{\bullet}_X) \to R\Gamma(X^{\mathrm{an}}, \Omega^{\bullet}_{X^{\mathrm{an}}}) \tag{1.7}$$

is an isomorphism. When composed with the inverse of (1.4), this yields an isomorphism

$$R\Gamma(X, \Omega^{\bullet}_X) \xrightarrow{\sim} R\Gamma(X^{\mathrm{an}}, \mathbb{C}) \tag{1.8}$$

or, after passing to cohomology, $H^*(X, \Omega^{\bullet}_X) \xrightarrow{\sim} H^*(X^{\mathrm{an}}, \mathbb{C})$. In the affine case, we have the equality $H^*(X, \Omega^{\bullet}_X) = H^*\Gamma(X, \Omega^{\bullet}_X)$, and so by the universal coefficient theorem we may conclude that

$$H^*\Gamma(X, \Omega^{\bullet}_X) \xrightarrow{\sim} \mathrm{Hom}(H_*(X^{\mathrm{an}}), \mathbb{C}), \tag{1.9}$$

where again the isomorphism is given explicitly by integrating differential forms along singular simplices in X. This is the corresponding version of de Rham's theorem for the case of schemes. Furthermore if X is instead a scheme over any field k with $\iota : k \hookrightarrow \mathbb{C}$ an embedding, and X_ι denotes the pullback of X along ι, recall that extension of scalars yields the canonical isomorphism

$$H^*(X, \Omega^{\bullet}_{X/k}) \otimes_k \mathbb{C} \xrightarrow{\sim} H^*(X_\iota, \Omega^{\bullet}_{X_\iota/\mathbb{C}}). \tag{1.10}$$

Combined with (1.8), this yields the period isomorphism analogous to (1.5) in this setting:

Theorem 3. *Let X, k, ι be as above. Then there is an isomorphism*

$$H^*(X, \Omega^{\bullet}_{X/k}) \otimes_k \mathbb{C} \xrightarrow{\sim} H^*(X_\iota^{an}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \tag{1.11}$$

depending on ι. Furthermore if X/k is proper and smooth, this isomorphism is compatible with the Hodge Filtration induced by the naive filtrations on the de Rham complexes.

For a complex algebraic variety X we may also consider *algebraic* differential forms [11]. As above, there are evident maps

$$H^q(X, \Omega^p_X) \to H^q(X^{\text{an}}, \Omega^p_{X^{\text{an}}}) \text{ and } H^*_{\text{dR}}(X) \to H_{\text{dR}}(X^{\text{an}}). \tag{1.12}$$

Furthermore these maps are isomorphisms in particular when X is projective, by Serre's GAGA theorem [20]. However, unlike the above cases, there is no analagous "algebraic" version of the Poincaré lemma that holds in this generality, so we are unable to construct a comparison isomorphism of the sort above. We will discuss this in more detail later on.

Artin [1] gave a way of describing the singular cohomology of X^{an} algebraically using the étale cohomology, whereby for $m > 1$ we have

$$H^*(X^{\text{an}}, \mathbb{Z}/m\mathbb{Z}) \xrightarrow{\sim} H^*_{\text{ét}}(X, \mathbb{Z}/m\mathbb{Z}), \tag{1.13}$$

and furthermore, taking the inverse limit of these isomorphisms for p a prime, this yields the isomorphism

$$H^*(X^{\text{an}}, \mathbb{Q}_p) \xrightarrow{\sim} H^*_{\text{ét}}(X, \mathbb{Q}_p). \tag{1.14}$$

But in general there is not a comparison with the de Rham cohomology. This is the motivation for moving to the p-adic setting, where the situation is clarified somewhat and constructing the desired comparison isomorphism is actually possible. The remainder of this chapter will detail the construction of a suitable isomorphism for comparing de Rham and étale cohomologies in the p-adic setting. The key ingredients will be devising the "correct" p-adic analog of the Poincaré lemma and construction of a suitable period ring to tensor with in our comparison. While the existence of such a comparison isomorphism has been known for some time (it was first conjectured by Fontaine in 1981 [7] and proved in general by Faltings in 1988 [6]), the most natural construction was given by Beilinson in 2011 [2], whose approach we will be following here. As shall be seen, Beilinson's techniques are quite geometric in nature, and don't require any of the syntomic cohomology or algebraic K-theory used by Fontaine. Indeed, Beilinson's approach is more or less in the spirit of the construction of Theorem 3.

1.2 The p-adic case, according to Beilinson

We will focus in particular on schemes over p-adic fields that are separated and of finite type, *i.e.* p-adic varieties. If k is a p-adic field with algebraic closure $\overline{k}$ and X is a variety over k, an isomorphism comparing the de Rham and étale cohomology theories will take the form

$$\rho : H^*_{\mathrm{dR}}(X) \otimes_k R \xrightarrow{\sim} H^*_{\text{ét}}(X, \mathbb{Z}_p) \otimes R, \tag{1.15}$$

where R is the some ring containing the periods analogous to the complex numbers obtained in the classical case. The appropriate R was originally constructed by Fontaine [9], and we will denote it in what followx by B_{dR}. Describing this ring will require some key technical notions; however, if we put aside for a moment the precise nature of B_{dR}, the broad strokes construction of the isomorphism ρ is strikingly similar to that of Theorem 3, again indicating that Beilinson's approach is the most natural.

Recall in particular the role of $\Omega^\bullet_{X^{\mathrm{an}}}$, the de Rham complex on X^{an}: the isomorphism in (1.8) factors through $R\Gamma(X^{\mathrm{an}}, \Omega^\bullet_{X^{\mathrm{an}}})$, inducing the period isomorphism in Theorem 3. The comparison has roughly the structure pictured in Figure 1.1.

$$\begin{array}{ccccc} & & R\Gamma(X^{\mathrm{an}}, \Omega^\bullet_{X^{\mathrm{an}}}) & & \\ & \nearrow & & \searrow & \\ R\Gamma(X, \Omega^\bullet_X) & & \dashrightarrow & & R\Gamma(X^{\mathrm{an}}, \mathbb{C})) \end{array}$$

Figure 1.1: The comparison pattern for varieties over $\mathbb{C}$

Describing the analogous object to "mediate" between $H^*_{\mathrm{dR}}(X) \otimes_{\overline{k}} B_{\mathrm{dR}}$ and $H^*_{\text{ét}}(X, \mathbb{Q}_p) \otimes B_{\mathrm{dR}}$ was one of the key insights of Beilinson. In this case, the object of interest in the factors of ρ is $\mathcal{A}^\natural_{\mathrm{dR}}$, a projective system of sheaves of differential graded algebras on $\mathrm{Var}_{\overline{k}}$, the category of $\overline{k}$-varieties. The basic structure of this comparison is pictured in Figure 1.2 – the rest of this section is devoted to understanding the maps and terms involved in this diagram.

$$\begin{array}{ccccc} \mathcal{A}^\natural_{\mathrm{dR}} \widehat{\otimes} \mathbb{Z}_p & \longleftarrow & \mathcal{A}^\natural_{\mathrm{dR}} & \longrightarrow & \mathcal{A}^\natural_{\mathrm{dR}} \otimes \mathbb{Q} \\ {\scriptstyle \varphi}\uparrow & & & & \downarrow{\scriptstyle \psi} \\ A_{\mathrm{dR}} \widehat{\otimes} \mathbb{Z}_p & & \overset{\rho}{\dashrightarrow} & & A_{\mathrm{dR}} \end{array}$$

Figure 1.2: The p-adic comparison pattern.

1.2.1 The derived de Rham algebra

We will now give some key technical definitions that enable us to formulate a p-adic version of the Poincaré lemma. Let R be a ring, and A an R-algebra. Following Illusie [13], we define the *cotangent complex* $L_{A/R}$ of A as

$$L_{A/R} = C(R_\bullet \otimes_{P_\bullet} \Omega^1_{P_\bullet/A}). \tag{1.16}$$

Here $R_\bullet$ is the constant simplicial object associated to R, $\epsilon_\bullet : P_\bullet \to R_\bullet$ is the standard simplicial resolution of R defined in [13], $\Omega^1_{P_\bullet}$ is the simplicial complex of $R-$modules $\Omega^1_{P_0} \to \Omega^1_{P_1} \to \Omega^1_{P_2} \dots$, and C is the associated chain complex. The cotangent complex is related to the algebraic de Rham complex according to the following proposition.

Proposition 1. *There is a natural isomorphism* $H_0(L_{A/R}) \xrightarrow{\sim} \Omega^1_{A/R}$.

Proof. As above, let $\epsilon_\bullet : P_\bullet \to R_\bullet$ denote an augmentation of $P_\bullet$. By definition of an augmentation map we have $\epsilon_0\delta_0 = \epsilon_0\delta_1$, where δ_n is the nth face map $\delta_n : P_n \to P_{n-1}$. Thus the composition in the associated chain complex C

$$A \otimes_{P_1} \Omega^{P_1/R} \to A \otimes_{P_0} \Omega^1_{P_0/R} \to \Omega^1_{A/R}$$

is zero. Hence there is a morphism of complexes $L_{A/R} \to \Omega^1_{A/R}$, where we are viewing the space $\Omega^1_{A/R}$ as a complex concentrated in degree 0. Since the ring homomorphism $P_0 \to A$ is surjective, so too is the induced map on homology $H_0(L_{A/R}) \to \Omega^1_{A/R}$. Letting $I = \ker(\epsilon_0 : P_0 \to B)$ we can use standard techniques in commutative algebra (see, for example, [16]) to obtain a short exact sequence

$$I/I^2 \to A \otimes_{P_0} \Omega^1_{P_0/R} \to \Omega^1_{A/R}.$$

Since the map $P_\bullet \to A$ is a resolution, the induced sequence is exact and thus we must have that $I = \mathrm{Im}(\delta_0 - \delta_1 : P_1 \to P_0)$. Thus the image of I/I^2 in $A \otimes_{P_0} \Omega^1_{P_0/R}$ is covered by $A \otimes_{P_1} \Omega^1_{P_1/R}$. This yields the desired result. □

Though we used the standard free resolution of R in the definition of the cotangent complex, the power of this definition is that the cotangent complex is invariant up to quasi-isomorphism with respect to other free resolutions. That is,

Theorem 4. *Let $Q_\bullet \to A$ be a simplicial resolution of an R-algebra A, whose terms are free*

$R-$algebras. Then there is a quasi-isomorphism of chain complexes of A-modules.

$$L_{A/R} \xrightarrow{\sim} C(R_\bullet \otimes_{Q_\bullet} \Omega^1_{Q_\bullet/A}) \tag{1.17}$$

We will not prove this here, as it requires some more commutative algebra machinery that won't be needed for the rest of this writing, but a detailed proof of this theorem may be found in [13]. Two important properties of the cotangent complex are the following.

Proposition 2. *If M and N are* Tor-independent R-algebras[1], then the natural base change morphism

$$R \otimes^L_R L_{N/R} \to L_{M \otimes_R N/M}$$

is actually a quasi-isomorphism of (complexes of) $M \otimes N$-modules.

Proposition 3. *A sequence $R \to S \to T$ of morphisms of rings induces an exact triangle in the derived category of complexes of T-modules:*

$$\begin{array}{ccc} T \otimes^L_S L_{S/R} & \longrightarrow & L_{T/R} \\ & \nwarrow \quad \swarrow & \\ & L_{T/S} & \end{array}$$

Consider now the double complex in Figure 1.3, obtained by applying the functor $R \mapsto \Omega^\bullet_R$ to the terms of $P_\bullet$.

$$\begin{array}{ccccccc} & & \vdots & & \vdots & & \vdots \\ & & \uparrow & & \uparrow & & \uparrow \\ \dots & \longrightarrow & \Omega^2_{P_2} & \longrightarrow & \Omega^2_{P_1} & \longrightarrow & \Omega^2_{P_0} \\ & & \uparrow & & \uparrow & & \uparrow \\ \dots & \longrightarrow & \Omega^1_{P_2} & \longrightarrow & \Omega^1_{P_1} & \longrightarrow & \Omega^1_{P_0} \\ & & \uparrow & & \uparrow & & \uparrow \\ \dots & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 \end{array}$$

Figure 1.3: The derived de Rham double complex.

The total complex associated with this double complex is called the *derived de Rham complex* or the *derived de Rham algebra* of R, and is denoted $L\Omega^\bullet_{A/R}$. Though $L\Omega^\bullet_{A/R}$ was defined only as a chain

[1] That is, if $\mathrm{Tor}^R_i(M,N) = 0$ for all $i > 0$.

complex, it is possible to equip it with a product structure induced by a product $P_i \otimes P_J \to P_{i+j}$ on $P_\bullet$. Endowing $L\Omega^\bullet_{A/R}$ with this multiplication in fact gives it the structure of a differential graded algebra over A, explaining the terminology derived de Rham *algebra*.

We can also endow $L\Omega^\bullet_{A/R}$ with a filtration compatible with this product structure. The *Hodge filtration* on $L\Omega^\bullet_{A/R}$ is induced by the naive filtration on the $\Omega^\bullet_{P_\bullet}$: we define $F^i(\Omega^\bullet_{P_\bullet/R}) = \Omega^{\geq i}_{P_\bullet/R}$. This is a filtration on the vertical chains, hence inducing a filtration on the total complex. We can *complete* the derived de Rham algebra with respect to this filtration, which will be very useful in practice. We define the *Hodge-completed derived de Rham algebra* as the projective system of chain complexes

$$L\widehat{\Omega}^\bullet_{A/R} = L\Omega^\bullet_{A/R}/F^i. \tag{1.18}$$

The graded pieces of the derived de Rham algebra are actually quite easily computable, as follows. This quasi-isomorphism also reveals much about the relationship between the derived de Rham algebra and Illusie's cotangent complex.

Proposition 4. *Let $P_\bullet$ denote the standard simplicial resolution of an R-algebra A. Then there is a quasi-isomorphism of chain complexes*

$$\mathrm{gr}^i_F \Omega^\bullet_{P_\bullet/R} \xrightarrow{\sim} L \wedge^i L_{A/R}[-i] \tag{1.19}$$

by which we may compute the graded pieces of $\Omega^\bullet_{P_\bullet/R}$ with respect to the Hodge filtration.

Proof. By definition, we have

$$\mathrm{gr}^i_F L\Omega^\bullet_{P_\bullet/R} \xrightarrow{\sim} \Omega^1_{P_\bullet/R}[-i] = (\cdots \to \Omega^i_{P_1}(R) \to \Omega^i_{P_0/R}).$$

We may view the augmentation $\epsilon : P_\bullet \to A_\bullet$ as a morphism of $P_\bullet$-modules, which induces a quasi-isomorphism between the associated chain complexes. A technical lemma of Illusie [13] yields a quasi-isomorphism

$$P_\bullet \otimes_{P_\bullet} \Omega^i_{P_\bullet/R} \xrightarrow{\sim} A_\bullet \otimes_{P_\bullet} \Omega^i_{P_\bullet/A},$$

due to the fact that $\Omega^i_{P_\bullet/R}$ a (simplicial) P-module with free terms. We compose this with the evident quasi-isomorphisms

$$A_\bullet \otimes_{P_\bullet} \Omega^i_{P_\bullet/A} \xrightarrow{\sim} A_\bullet \otimes_{P_\bullet} \wedge^i \Omega^1_{P_\bullet/A} \xrightarrow{\sim} \wedge^i (A_\bullet \otimes_{P_\bullet} \Omega^1_{P_\bullet/R}),$$

yielding the desired quasi-isomorphism.

□

This proposition allows us to formulate an analog of Theorem 4 for the derived de Rham algebra, allowing us to define it up to quasi-isomorphism using *any* simplicial resolution of A.

Theorem 5. *Let $Q_\bullet \to A$ be a simplicial resolution of A with terms that are free R-algebras. Then there is a quasi-isomorphism of chain complexes*

$$L\Omega^\bullet_{A/R} \xrightarrow{\sim} \mathrm{Tot}(\Omega^\bullet_{Q_\bullet/R}). \tag{1.20}$$

Furthermore, this quasi-isomorphism is compatible with the Hodge filtration and the product structure on $L\Omega^\bullet_{(A/R)}$.

Proof. See [13] □

1.2.2 The h-topology and the comparison isomorphisms

We are working toward Fontaine's period ring B_{dR}. As before let k be a p-adic field with algebraic closure $\overline{k}$. Following Beilinson [2] we define $A_{\mathrm{dR}} := L\widehat{\Omega}^\bullet_{\mathcal{O}_{\overline{k}}/\mathcal{O}_k}$, which is a projective system of $\mathcal{O}_{\overline{k}}$ modules equipped with a product structure and a filtration. Fontaine [9] defines a ring B^+_{dR} as follows ([13], [2]), which does not depend on k and will turn out to be the (complete) valuation ring of B_{dR}, Fontaine's ring of p-adic periods.

$$B^+_{\mathrm{dR}} := \varprojlim_i (((\mathcal{O}_k \otimes_W W(\varprojlim_{x\mapsto x^p} \mathcal{O}_{\overline{k}}/p))/(\ker\theta)^i)^\wedge \otimes \mathbb{Q}), \tag{1.21}$$

with $W = W(k)$ the ring of Witt vectors over k [17], $\theta : \mathcal{O}_k \otimes_W W(\varprojlim_{x\mapsto x^p} \mathcal{O}_{\overline{k}}/p) \to \mathcal{O}_{\widehat{\overline{k}}}$ is the canonical map sending $(x_0, x_1, \ldots, x_n, \ldots) \in W(\varprojlim_{x\mapsto x^p} \mathcal{O}_{\overline{k}}/p)$ to $\sum p^n x_n(n)$, where $x_n^{(m)} \in \mathcal{O}_{\overline{k}}$, $(x_n^{(m+1)})^p = x_n^{(m)}$, and $^\wedge$ denotes the p-adic completion. Though this definition seems quite removed from the present situation, a result of Beilinson [2] relates it neatly to A_{dR}.

Proposition 5. *There is a canonical isomorphism*

$$B^+_{\mathrm{dR}} \xrightarrow{\sim} R\varprojlim_i (((A_{\mathrm{dR}}/F^i)\widehat{\otimes}\mathbb{Z}_p) \otimes \mathbb{Q}), \tag{1.22}$$

where $\widehat{\otimes}$ is the completed derived tensor product, as used by Beilinson: for a complex E of abelian *groups, define*

$$E\widehat{\otimes}\mathbb{Z}_p = R\varprojlim_i (E \otimes^L \mathbb{Z}/p^i), \tag{1.23}$$

where $\otimes^L$ *is the derived tensor product.* [2]

Proof. I will provide only a sketch of the proof, based on Illusie [14]: see [2] for a more detailed treatment. Fontaine [7] gives the relation

$$\Omega^1_{\mathcal{O}_{\overline{k}}/\mathcal{O}_k} = (\overline{k}/\mathfrak{a})(1),$$

where $\mathfrak{a}$ is defined as the fractional ideal of $\mathcal{O}_{\overline{k}}$ generated by $p^{-1/(p-1)}\mathcal{D}^{-1}_{k/k_o}$, with k_0 the fraction field of $W(k)$ and $\mathcal{D}_{k/k_0}$ is the different. That is, $\mathfrak{a} = p^{-1/(p-1)}\mathcal{D}^{-1}_{k/k_0} \cdot \mathcal{O}_{\overline{k}} \subset \overline{k}$. Hence we have canonical isomorphisms

$$L_{\mathcal{O}_{\overline{k}}/\mathcal{O}_k} \xrightarrow{\sim} \Omega^1_{\mathcal{O}_{\overline{k}}/\mathcal{O}_k} \xrightarrow{\sim} (\overline{k}/\mathfrak{a})(1) \simeq (\mathbb{Q}_p/\mathbb{Z}_p) \otimes \mathfrak{a}(1).$$

We can apply Quillen's shift formula [13] for M an A-module to obtain, for all nonnegative i,

$$L \wedge^i (M[1]) \xrightarrow{\sim} L\Gamma^i(M)[i].$$

Therefore we can calculate the cohomology of the graded objects directly, as

$$H^n(\mathrm{gr}_F A_{\mathrm{dR}}\widehat{\otimes}\mathbb{Z}_p) \begin{cases} 0 & \text{if} n \neq 0 \\ \widehat{\mathcal{O}}_{\overline{k}}\langle \widehat{\mathfrak{a}}(1)\rangle & \text{else,} \end{cases}$$

where $\langle \bullet \rangle$ denotes a divided power algebra [18]. The point is that $(A_{\mathrm{dR}}/F^{i+1})\widehat{\otimes}\mathbb{Z}_p$ is concentrated in degree $n = 0$, with

$$((A_{\mathrm{dR}}/F^{i+1})\widehat{\otimes}\mathbb{Z}_p) \otimes \mathbb{Q} \xrightarrow{\sim} C[t]/t^{i+1}.$$

By definition we have $A_{\mathrm{dR}}/F^2 = (\mathcal{O}_{\overline{k}} \xrightarrow{d} \Omega^1_{\mathcal{O}_{\overline{k}}/\mathcal{O}_k})$, so that $A_{\mathrm{dR}}/F^2 \xrightarrow{\sim} \ker d =: \mathcal{O}'_{\overline{k}}$, since d is surjective [8]. Let $A_{\mathrm{inf}} = \varprojlim_i ((\mathcal{O}_k \otimes_{W(k)} W(\varprojlim_{x\mapsto x^p} \mathcal{O}_{\overline{k}}/p))/\ker\theta^i)^\wedge$ be the universal thickening (see [7]). Define the collection of maps

$$u_i : A_{\mathrm{inf}}/F^{i+1} \to (A_{\mathrm{dR}}/F^{i+1})\widehat{\otimes}\mathbb{Z}_p.$$

[2]The derived tensor product is the left derived functor of the tensor product functor $\cdot\otimes\cdot : \mathrm{Mod}_A \times {}_A\mathrm{Mod} \to {}_R\mathrm{Mod}$. More precisely, we have that $\cdot \otimes^L_A \cdot : D(\mathrm{Mod}_A) \times D({}_A\mathrm{Mod}) \to D({}_R\mathrm{Mod})$ is the derived tensor product functor, for $D(\mathrm{Mod}_A)$ and $D({}_A\mathrm{Mod})$ the derived categories associated to the categories of right and left A-modules, respectively, for A a dga over a ring R.

Again, Fontaine showed in [7] that $u_1 : A_{\mathrm{inf}}/F^2 \to (A_{\mathrm{dR}}/F^2)\widehat{\otimes}\mathbb{Z}_p$ is a filtered isomorphism, so that

$$u_{i\mathbb{Q}} : B^+_{\mathrm{dR}}/F^{i+1} \to (A_{\mathrm{dR}}/F^{i+1})\widehat{\otimes}\mathbb{Q}_p := ((A_{\mathrm{dR}}/F^{i+1})\widehat{\otimes}\mathbb{Z}_p) \otimes \mathbb{Q}$$

is a filtered isomorphism as well. Taking inverse limits yields the desired isomorphism of the proposition. □

Accordingly, we define B_{dR}, also known as the Fontaine period ring, as the fraction field of the discrete valuation ring B^+_{dR}.

The goal is now to sheafify A_{dR} with respect to a Grothendieck topology on the category of schemes fine enough to trivialize higher cohomology of these complexes over small open sets, which will furnish a p-adic Poincaré lemma. While this notion seems very natural when viewed next to *e.g.* Lemma 2, actually finding a suitable Grothendieck topology was one of the key insights of Beilinson.

Note that neither the proper topology, generated by proper and surjective coverings of a scheme S, nor the étale topology, generated by standard étale coverings of S, is fine enough for our purposes [2]. Instead, we turn to Voevodsky's *h-topology*, with covering families generated by both proper surjective maps and étale surjective maps. More precisely, Voevodsky define the h-topology as follows [19].

A morphism of schemes $p : X \to Y$ is called a *topological epimorphism* if the induced map on underlying Zariski topological spaces is a quotient map, in the sense that p is surjective and the topology on Y coincides with the quotient topology induced by p. Furthermore, such a map is *universal topological epimorphism* if, for any Z/Y, the base-change morphism $p_Z : X \times_Y Z\ toZ$ is a topological epimorphism. These maps generate a covering of a scheme X: an $h-$covering of X is a finite family of morphisms of finite type $p_i : X_i \to X$ such that $\coprod p_i : \coprod X_i \to X$ is a universal topological morphism. These h-coverings define a pre-topology on the category of schemes, and the *h-topology* is the associated topology. Notice in particular that the h-topology is finer than both the proper and étale topologies, as desired.

This use of the h-topology is inspired by a theorem of Bhatt [4]. Roughly, each higher cohomology class of a coherent sheaf is p-divisible after passing to an appropriate proper and surjective covering or, in this case, after tensoring with $\mathbb{Z}_p$. This indicates that we should consider the tensor product $A_{\mathrm{dR}}\widehat{\otimes}\mathbb{Z}_p$ rather than A_{dR} in order for the higher cohomology of $A_{\mathrm{dR}}\widehat{\otimes}\mathbb{Z}_p$ to vanish on small open sets. The idea is to write $\mathcal{A}^{\natural}_{\mathrm{dR}}$ to denote the sheafification of A_{dR} with respect to the h-topology (the precise meaning of this will be made clear), allowing us to state Beilinson's p-adic Poincaré lemma [2] [20]:

Theorem 6 (p-adic Poincaré lemma)**.** *The maps*

$$A_{\mathrm{dR}}\widehat{\otimes}\mathbb{Z}_p \to \mathcal{A}^{\natural}\widehat{\otimes}\mathbb{Z}_p \tag{1.24}$$

are quasi-isomorphisms (where A_{dR} is viewed as a constant h-sheaf), compatible with the Hodge filtration.

Note that this is the map φ in Figure 1.2. As a result, we have the following

Corollary 1. *Let X be a smooth variety over a field k. Then for each $n \geq 0$ there is a filtered isomorphism*

$$H^n_{\text{ét}}(X_{\overline{k}}, \mathbb{Z}) \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}} \xrightarrow{\sim} H^n_h(X_{\overline{k}}, \mathcal{A}^{\natural}_{\mathrm{dR}})\widehat{\otimes}\mathbb{Q}_p. \tag{1.25}$$

The goal is to use this h-sheafification as a "bridge" to modulate between étale and de Rham cohomology on X. Therefore, to construct the p-adic comparison isomorphism we must understand how the right hand side of 1.25 relates to the de Rham cohomology of X. Indeed, Beilinson does this by proving that $\mathcal{A}^{\natural} \otimes \mathbb{Q}$ is precisely the h-sheafification of the Hodge-completed logarithmic de Rham complex.[2] Formalizing these ideas requires some finesse. I will summarize several of the technical difficulties that arise – for a more complete picture, see [2] [20].

To ensure that the de Rham complexes in question are well-behaved, we work only in the generality of smooth varieties U over a characteristic zero field k with a smooth normal crossing compactification $\overline{U}$, *i.e.* such that there is a smooth compactification $j : U \to \overline{U}$ such that the divisor $D = \overline{U}/U$ has normal crossings. A suitable compactification exists due to Hironaka's theorem on resolution of singularities [12].[3]

With U, $\overline{U}$, D, and j as above, we define the *logarithmic de Rham complex* $\Omega^{\bullet}_{\overline{U}/k}(\log D)$ as the subcomplex of $j_*\Omega^{\bullet}_{U/k}$ for which the terms have local sections $w \in j_*\Omega^i_{U/k}(V)$ such that both ω and $d\omega$ have at worst logarithmic singularities along D, for V a small open set. A more thorough treatment of these complexes may be found in [21].

Denote by $\mathcal{P}_k$ the category of such pairs $(U, \overline{U})$, *i.e.* where U is a smooth k-variety, $\overline{U}$ is a smooth compactification, and $\overline{U} \setminus U$ is a normal crossings divisor. The contravariant functor on $\mathcal{P}_k$ sending these pairs to the associated logarithmic de Rham complexes

[3]Namely, Hironaka's resolution of singularities ensures that a smooth variety X may be viewed as an open variety of some smooth projective variety such that the boundary is a normal crossings divisor.

$$(U,\overline{U}) \mapsto \Omega^\bullet_{\overline{U}/k}(\log D). \tag{1.26}$$

This functor is a presheaf on $\mathcal{P}_k$. We cannot just naively sheafify the total derived functor of this presheaf, as it takes values in a derived category and wouldn't yield a sheaf on $\mathcal{P}_k$. However, Illusie [13] uses Godement resolutions (see [21]) to find a complex $C^\bullet$ representing this functor, yielding the presheaf

$$(U,\overline{U}) \mapsto \Gamma(U, C^\bullet(\Omega^\bullet_{\overline{U}/k}(\log D)), \tag{1.27}$$

which is more or less a derived version of the above presheaf. We may pullback the h-topology along the forgetful functor $\mathcal{P}_k \to \mathrm{Var}_k$ to sheafify 1.27 into an h-sheaf on $\mathcal{P}_k$. We denote the h-sheaf thus obtained by $\mathcal{A}_{\mathrm{dR}}$. Note that the Hodge filtration on $\Omega^\bullet_{\overline{U}/k}(\log D)$ induces a filtration on $\mathcal{A}_{\mathrm{dR}}$. In fact, $\mathcal{A}_{\mathrm{dR}}$ yields an h-sheaf on the category Var_k as well:

Theorem 7. *There is an equivalence of categories of h-sheaves over $\mathcal{P}_{\overline{k}}$ and h-sheaves over $\mathrm{Var}_{\overline{k}}$ induced by the forgetful functor $\mathcal{P}_{\overline{k}} \to \mathrm{Var}_{\overline{k}}$.*

Proof. The proof requires some sheaf-theoretic preliminaries. We first recall a classical comparison result for Grothendieck topologies, due to Verdier.

Lemma 3. *Let $F : C \to C'$ be a functor between small categories. Let C' be equipped with a Grothendieck topology, and C equipped with the induced Grothendieck topology (the finest topology in which sheaves on C' pullback to sheaves on C). If F is fully faithful and every object of C' has a covering by objects in the image of F, then the pullback functor induces an equivalence of the category of sheaves on C' with the category of sheaves on C.*

In other words, under a relatively unrestrictive set of conditions, we can construct a push-forward functor of sheaves on C to sheaves on C' which is right adjoint to the pullback functor.

Beilinson refined Lemma 3 to account for functors which are faithful but not fully faithful by replacing the covering condition in Lemma 3 by the following, more complicated, condition:

Condition (∗): For every $V \in C'$ and a finite family of pairs (W_α, f_α) with $W_\alpha \in C$ and $f_\alpha : V \to F(W_\alpha)$ morphisms in C', there exists a set of objects $W_\beta \in C$ together with morphisms $F(W_\beta) \to V$ in C' satisfying:

- The morphisms $F(W_\beta) \to V$ form a covering family of V.
- Every composite morphism $F(W_\beta) \to V \to F(W_\alpha)$ is in the image of a morphism $W_\beta \to W_\alpha$ via F.

With this refined condition, Beilinson gave the following analogue of Lemma 3

Lemma 4. *If C, C' are as in the previous lemma and $F : C \to C'$ is a faithful functor satisfying condition $(*)$, then the pullback functor induces an equivalence of the category of sheaves on C' with the category of sheaves on C for the topology induced by F.*

Note that Beilinson's condition reduces to Verdier's in the case where the B_α are an empty set. We can now prove Theorem 7.

Apply Lemma 4 in the situation where C' is Var_k equipped with the h-topology, and F is the (faithful) forgetful functor $(V, \overline{V}) \mapsto V$ from the category of $\mathcal{P}'_k$ of such pairs where V is a k-variety and $\overline{V}$ is a proper k-variety containing V as a dense open subset. Nagata's theorem ensures that condition $(*)$ is satisfied in this case.

Since the inclusion functor $\mathcal{P}_k \to \mathcal{P}'_k$ is fully faithful, we may apply Lemma 3. It is left to check that each pair $(V, \overline{V})$ in $\mathcal{P}'_k$ has an h-covering $(U, \overline{U}) \to (V, \overline{V})$ by a pair in $\mathcal{P}_k$, which follows from Hironaka's theorem.

□

For any smooth k-variety X, this morphism of complexes of presheaves $C^\bullet(\Omega^\bullet_{\overline{U}/k}(\log D)) \to \mathcal{A}_{\mathrm{dR}}$ induces morphisms

$$R\Gamma_{\mathrm{dR}}(X/k) \to R\Gamma_h(X, \mathcal{A}_{\mathrm{dR}}). \tag{1.28}$$

Theorem 8. *Let X be a smooth k-variety. Then the maps appearing in 1.28 are filtered quasi-isomorphisms.*

Proof. We may assume that $k = \mathbb{C}$, by a Lefschetz principle argument. Let $\overline{X}$ be a smooth normal crossing compactification of X with complement D. From Beilinson, there exists an h-hypercovering $V_\bullet \to X$ such that each V_n is a msooth k-scheme of finite type, and furthermore there is a simplicial compactification $V_\bullet \hookrightarrow \overline{V}_\bullet \hookrightarrow \overline{Y}$ such that $\overline{V}_n$ is proper and smooth with $D_n := \overline{V}_n \setminus V_n$ a normal crossing divisor. On $V_\bullet$, consider the simplicial complex of presheaves $\Omega^\bullet_{\overline{V}_\bullet/k}(\log D_\bullet)$.

By [10], there is a filtered quasi-isomorphism

$$R\Gamma(\overline{V}_\bullet, \Omega^\bullet_{\overline{V}_\bullet \setminus \mathbb{C}}(\log D_\bullet)) \simeq R\Gamma_{\mathrm{sing}}(\overline{V}_\bullet, \mathbb{C}),$$

where the RHS is complex singular cohomology. Likewise there is a filtered quasi-isomorphism

$$R\Gamma(\overline{Y}_\bullet, \Omega^\bullet_{\overline{Y}_\bullet \setminus \mathbb{C}}(\log D_\bullet)) \simeq R\Gamma_{\mathrm{sing}}(\overline{Y}_\bullet, \mathbb{C})$$

.

These isomorphisms induce commutative diagrams on the level of cohomology for all dimensions n, and again by [10] we obtain a vertical isomorphism

$$H^n_{\mathrm{sing}}(\overline{V}_\bullet, \mathbb{C}) \simeq H^n_{\mathrm{sing}}(\overline{Y}, \mathbb{C}),$$

hence by commutativity the following is also an isomorphism:

$$\mathbb{H}^n(\overline{V}_\bullet, \Omega^\bullet_{\overline{V}_\bullet/\mathbb{C}}(\log D_\bullet)) \simeq \mathbb{H}^n(\overline{Y}, \Omega^\bullet_{\overline{Y}/\mathbb{C}}(\log D)).$$

Recall that $R\Gamma(\overline{Y}, \Omega^\bullet_{\overline{Y}/\mathbb{C}}(\log D))$ is computed in the Zariski topology by $\Gamma(\overline{Y}, C^\bullet(\Omega^\bullet_{(Y,\overline{Y})/\mathbb{C}}))$, and similarly for the simplicial version. Hence that the direct system $H^n(\overline{V}_\bullet, C^\bullet\Omega^\bullet_{\overline{V/\mathbb{C}}}(\log D))$ is constant for all $V_\bullet$ as above. The direct limit of this system is $H^n(X, \mathcal{A}_{\mathrm{dR}})$, so we are done.

□

For reasons that will become clear when we construct the comparison isomorphism, it is useful to formulate a "Hodge-completed" analog of the above theorem. The *Hodge-completed de Rham complex* $\widehat{\Omega}^\bullet_{\overline{U}/k}(\log D)$ is defined to be the inverse system of the quotients $\Omega^\bullet_{\overline{U}/k}(\log D)/F^i$, with F^i denoting the Hodge filtration. In parallel to above, we write $\widehat{\mathcal{A}}_{\mathrm{dR}}$ as the h-sheaf associated to the presheaf

$$(U, \overline{U}) \mapsto \Gamma(\overline{U}, C^\bullet(\widehat{\Omega}^\bullet_{\overline{U}/k}(\log D)))$$

on the category $\mathcal{P}_k$. Again by Theorem 7, this sheafifies to a sheaf on Var_k, yielding morphisms

$$R\Gamma_{\mathrm{dR}}(X)^\wedge \to R\Gamma_h(X, \widehat{\mathcal{A}}_{\mathrm{dR}}). \tag{1.29}$$

The situation here is much the same as for the non-completed complex. Since the Hodge filtration on any fixed H^n_{dR} is finite, the cohomology groups of the left hand side of the above coincide with those of the non-completed complex. So, as a corollary of Theorem 8, these morphisms are also filtered quasi-isomorphisms.

Next, consider when k is a finite extension of $\mathbb{Q}_p$, *i.e.* when k is a p-adic field. We define a *semistable pair over* k to be a pairing $(U,\mathcal{U})$ comprising a smooth k-variety U along with an open embedding $j : U \to \mathcal{U}$ with dense image into a reduced proper flat $\mathcal{O}_k$-scheme $\mathcal{U}$ and $D = \mathcal{U}/U$ a normal crossings divisor. Then define a *semistable pair over* $\overline{k}$ to be a pair $(U,\mathcal{U})$ defined by an open immersion of a $\overline{k}$-variety U in a flat proper $\mathcal{O}_{\overline{k}}$ scheme $\mathcal{U}$, arising via base change from a semistable pair (in the above sense) $(U', \mathcal{U}')$ defined over a finite extension k'/k. We denote by $\mathcal{SS}_{\overline{k}}$ the category of semistable pairs over $\overline{k}$.

Let $(V, \mathcal{V})$ be a semistable pair over $\overline{k}$. Beilinson uses techniques from logarithmic geometry (a good basic reference is [15]) to show the existence of the (completed) *derived* log de Rham algebra $L\Omega^{\bullet}_{(V,\mathcal{V})/\mathcal{O}_k}$, defined analogously to the usual derived de Rham algebra. Now, consider the contravariant functor on $\mathcal{SS}_{\overline{k}}$ given by the association

$$(V, \mathcal{V}) \mapsto \Gamma(\mathcal{V}, L\widehat{\Omega}^{\bullet}_{(V,\mathcal{V})/\mathcal{O}_k}). \tag{1.30}$$

By applying the Godement resolution $C^{\bullet}$ for the Zariski topology to the associated derived functor $R\Gamma(\mathcal{V}, L\widehat{\Omega}^{\bullet}_{(V,\mathcal{V})/\mathcal{O}_k})$, we obtain a presheaf [21] on $\mathcal{SS}_{\overline{k}}$:

$$(V, \mathcal{V}) \mapsto \Gamma(\mathcal{V}, C^{\bullet}(L\widehat{\Omega}^{\bullet}_{(V,\mathcal{V})/\mathcal{O}_k})), \tag{1.31}$$

sending semistable pairs over $\overline{k}$ to projective systems of complexes of the sheaves $\Gamma(\mathcal{V}, C^{\bullet}(L\Omega^{\bullet}_{(V,\mathcal{V}/\mathcal{O}_k)}/F^i))$.

Now consider the forgetful functor $\mathcal{SS}_{\overline{k}} \to \mathrm{Var}_{\overline{k}}$ from the category of semistable pairs over k to the category of k-varieties. We pullback the h-topology over $\mathrm{Var}_{\overline{k}}$ to $\mathcal{SS}_k$ along the forgetful functor and sheafify the presheaf 1.31 to obtain an h-sheaf over $\mathcal{SS}_{\overline{k}}$ which we will denote by $\mathcal{A}^{\natural}_{\mathrm{dR}}$. In fact, this also induces an h-sheaf over $\mathrm{Var}_{\overline{k}}$ according to the following theorem, which is the analog of Theorem 8 in this setting.

Theorem 9. *There is an equivalence of categories of h-sheaves over $\mathcal{SS}_{\overline{k}}$ and h-sheaves over $\mathrm{Var}_{\overline{k}}$ induced by the forgetful functor $\mathcal{SS}_{\overline{k}} \to \mathrm{Var}_{\overline{k}}$.*

With these notions in hand, we are finally able to construct the map ψ to complete the comparison pattern in Figure 1.2, following the construction of [20]. We begin with the following proposition of Beilinson [2]

Proposition 6. *There is a canonical isomorphism*

$$\mathcal{A}^{\natural}_{\mathrm{dR}} \otimes \mathbb{Q} \xrightarrow{\sim} \widehat{\mathcal{A}}_{\mathrm{dR}}. \tag{1.32}$$

With this in mind, consider the morphisms of log schemes

$$(U, \mathcal{U}) \xrightarrow{f} \mathrm{Spec}\mathcal{O}_{\overline{k} \to \mathcal{O}_{\overline{k}}},$$

where the second and third terms have trivial log structure. By 5, we obtain from this sequence a transitivity triangle of log cotangent complexes, which induces a map of cotangent complexes

$$f^*L_{\mathcal{O}_{\overline{k}}/\mathcal{O}_k} \to L_{(U,\mathcal{U})/\mathcal{O}_k}$$

and a map of derived log de Rham complexes

$$f^*L^\bullet_{\mathcal{O}_{\overline{k}}/\mathcal{O}_k} \to L\Omega^\bullet_{(U,\mathcal{U})/\mathcal{O}_k}. \tag{1.33}$$

We can quotient by the filtration F^i to identify the left hand side of (1.33) with the constant sheaf on U associated to A_{dR}/F^i. As discussed earlier, the higher cohomologies of this sheaf are trivial, so we obtain the following morphism of complexes

$$A_{\mathrm{dR}}/F^i \to \Gamma(\mathcal{U}, C^\bullet(L\Omega^\bullet_{U,\mathcal{U}/\mathcal{O}_k}/F^i)), \tag{1.34}$$

where A_{dR} is viewed as a complex concentrated in degree 0. After sheafifying the right hand side with respect to the h-topology, we obtain morphisms

$$A_{\mathrm{dR}}/F^i \to \mathcal{A}^\natural_{\mathrm{dR}}/F^i. \tag{1.35}$$

1.2.3 Constructing the comparison isomorphism.

We are now ready to return to Beilinson's version of the p-adic Poincaré lemma:

Theorem 6 (p-adic Poincaré lemma)**.** *The maps*

$$A_{\mathrm{dR}}/F^i\widehat{\otimes}\mathbb{Z}_p \to \mathcal{A}^\natural_{\mathrm{dR}}/F^i\widehat{\otimes}\mathbb{Z}_p \tag{1.36}$$

induced by 1.35 are filtered quasi-isomorphisms, for all i.

Taking this theorem for granted for the moment, we have the following corollary [20]

Corollary 2. *Let X be a smooth k-variety with a smooth normal crossing compactification. Then there are filtered quasi-isomorphisms*

$$R\Gamma_{\text{ét}}(X_{\overline{k}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} (B^+_{\mathrm{dR}}/F^i) \xrightarrow{\sim} R\Gamma_h(X_{\overline{k}}, \mathcal{A}^{\natural/F_i}_{\mathrm{dR}})\widehat{\otimes}\mathbb{Q}_p$$

for all i. Hence in the limit we obtain the filtered quasi-isomorphism

$$R\Gamma_{\text{ét}}(X_{\overline{k}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B^+_{\text{dR}} \xrightarrow{\sim} R\Gamma_h(X_{\overline{k}}, \mathcal{A}^{\natural}_{\text{dR}})\widehat{\otimes}\mathbb{Q}_p.$$

To prove this, we first need the following proposition, for which proof can be found in [20]

Proposition 7. *Let X be a reduced connected noetherian excellent scheme, and A a torsion abelian group. If $A_{\text{ét}}$ and A_h are the constant étale and h-sheaves on X, respectively, there is a canonical quasi-isomorphism*

$$R\Gamma(X_{\text{ét}}, A_{\text{ét}}) \xrightarrow{\sim} R\Gamma(X_h, A_h)$$

Proof. (Corollary 2) The following maps are quasi-isomorphisms for each i, by definition of the derived tensor product.

$$R\Gamma_{\text{ét}}(X_{\overline{k}}, \mathbb{Z}_p) \otimes^L_{\mathbb{Z}_p} (A_{\text{dR}}/F^i) \xrightarrow{\sim} R\Gamma_{\text{ét}}(X_{\overline{k}}, A_{\text{dR}}/F^i). \tag{1.37}$$

Note that the completed tensor product introduced earlier is an exact functor, so by taking the complete tensor with $\mathbb{Z}_p$ we obtain quasi-isomorphisms

$$R\Gamma_{\text{ét}}(X_{\overline{k}}, \mathbb{Z}_p) \otimes^L_{\mathbb{Z}_p} (A_{\text{dR}}/F^i)\widehat{\otimes}\mathbb{Z}_p \xrightarrow{\sim} R\Gamma_{\text{ét}}(X_{\overline{k}}, (A_{\text{dR}}/F^i)\widehat{\otimes}\mathbb{Z}_p),$$

and furthermore applying the above proposition yields quasi-isomorphisms

$$R\Gamma_{\text{ét}}(X_{\overline{k}}, (A_{\text{dR}}/F^i)\widehat{\otimes}\mathbb{Z}_p) \xrightarrow{\sim} R\Gamma_h(X_{\overline{k}}, (A_{\text{dR}}/F^i)\widehat{\otimes}\mathbb{Z}_p).$$

Now we compose with the quasi-isomorphism of Theorem **??** to obtain

$$R\Gamma_{\text{ét}}(X_{\overline{k}}, (A_{\text{dR}}/F^i)\widehat{\otimes}\mathbb{Z}_p) \xrightarrow{\sim} R\Gamma_h(X_{\overline{k}}, (\mathcal{A}^{\natural}_{\text{dR}}/F^i)\widehat{\otimes}\mathbb{Z}_p). \tag{1.38}$$

Combining this chain of quasi-isomorphisms and again making use of the fact that $\cdot\widehat{\otimes}\mathbb{Z}_p$ is an exact functor,

$$R\Gamma_{\text{ét}}(X_{\overline{k}}, \mathbb{Z}_p) \otimes^L_{\mathbb{Z}_p} (A_{\text{dR}}/F^i)\widehat{\otimes}\mathbb{Z}_p \xrightarrow{\sim} R\Gamma_h(X_{\overline{k}}, (\mathcal{A}^{\natural}_{\text{dR}}/F^i)\widehat{\otimes}\mathbb{Z}_p)\widehat{\otimes}\mathbb{Z}_p.$$

The result then follows by applying 5 and tensoring with $\mathbb{Q}$. ∎

With the p-adic Poincaré lemma in hand, we can explicitly construct the promised comparison map

$$\rho : H^*_{\mathrm{dR}}(X) \otimes_k B_{\mathrm{dR}} \xrightarrow{\sim} H^*_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}, \tag{1.39}$$

where as before X is a smooth k-variety with a smooth normal crossing compactification. We compose the quasi-isomorphisms of (1.29) and (1.32) to obtain

$$R\Gamma_{\mathrm{dR}}(X)^{\wedge} \xrightarrow{\sim} R\Gamma_h(X_{\overline{k}}, \mathcal{A}^{\natural}_{\mathrm{dR}}) \otimes \mathbb{Q}. \tag{1.40}$$

Consider the obvious map

$$R\Gamma_h(X_{\overline{k}}, \mathcal{A}^{\natural}_{\mathrm{dR}}) \mapsto R\Gamma_h(X_{\overline{k}}, \mathcal{A}^{\natural}_{\mathrm{dR}}) \widehat{\otimes} \mathbb{Z}_p.$$

We can tensor this map with $\mathbb{Q}$ and compose with 1.40 to obtain

$$R\Gamma_{\mathrm{dR}}(X_{\overline{k}})^{\wedge} \to R\Gamma_h(X_{\overline{k}}, \mathcal{A}^{\natural}_{\mathrm{dR}}) \widehat{\otimes} \mathbb{Q}_p.$$

Applying Corollary 2 to this map yields

$$R\Gamma_{\mathrm{dR}}(X_{\overline{k}})^{\wedge} \to R\Gamma_{\text{ét}}(X_{\overline{k}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B^+_{\mathrm{dR}}.$$

Now we can just tensor up with B^+_{dR} and pull back along the map $R\Gamma_{\mathrm{dR}}(X)^{\wedge} \to R\Gamma_{\mathrm{dR}}(X_{\overline{k}})^{\wedge}$ to obtain

$$R\Gamma_{\mathrm{dR}}(X)^{\wedge} \otimes_k B^+_{\mathrm{dR}} \to R\Gamma_{\text{ét}}(X_{\overline{k}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B^+_{\mathrm{dR}}.$$

The desired comparison isomorphism

$$\rho : H^*_{\mathrm{dR}}(X) \otimes_k B_{\mathrm{dR}} \xrightarrow{\sim} H^*_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}} \tag{1.41}$$

then falls out after passing to the field of fractions B_{dR} and taking cohomology. Thus we have constructed the p-adic de Rham comparison isomorphism.

To prove that 1.41 is actually an isomorphism, Beilinson performs a computation specifically for the case $X = \mathbb{G}_m = \mathrm{Spec} k[x, x^-1]$, followed by what he terms "usual tricks of the trade" to show the isomorphism holds in general. Indeed, since $\mathbb{G}_m$ is connected and of dimension 1, the only isomorphism necessary to verify is for the case $n = 1$:

$$\rho : H^1_{\mathrm{dR}}(\mathbb{G}_m) \otimes_k B_{\mathrm{dR}} \to H^1_{\text{ét}}(\mathbb{G}_m, \mathbb{Z}_p) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}. \tag{1.42}$$

Beilinson performs this verification explicitly by computing the relevant cohomology theories, before exploiting de Jong's alteration techniques [5] to generalize prove the theorem in full generality.

Chapter 2

Spectral sequences and Filtrations

We will conclude by approaching this problem from a slightly different angle. In the last chapter we described the derived de Rham cohomology of varieties over p-adic fields, but we did not provide a practical means of actually computing this cohomology, nor of computing algebraic de Rham cohomology. Work by Bhatt [3] has more or less cracked the problem of computing the algebraic de Rham cohomology of algebraic varieties in characteristic 0, but there is some interesting work left to do in understanding the structure of the filtrations and spectral sequences that arise in this direction.

2.1 Complex manifolds

We will first recall some facts about the Hodge-to-de-Rham spectral sequence (also called the *Frölicher spectral sequence*), used to compute the usual de Rham cohomology of complex manifolds. For a more detailed treatment of this subject, refer to [21]. Let X be a complex manifold with de Rham complex $\Omega_X^\bullet$. Then the Poincaré lemma 2 says that the sequence

$$0 \to \mathbb{C} \to \mathcal{O}_x \to \Omega_X^1 \to \dots \tag{2.1}$$

is exact, so that the de Rham cohomology is trivial over contractible open subsets. This induces an isomorphism

$$H^n(X, \mathbb{C}) \xrightarrow{\sim} \mathbb{H}^n(X, \Omega_X^\cdot) = H^n_{\mathrm{dR}}(X), \tag{2.2}$$

where $\mathbb{H}^n(X, \Omega_X^\cdot$ denotes the hypercohomology of the complex $\Omega_X^\bullet$. Recall also the *naive filtration* on $\Omega_X^\bullet$, the filtration obtained by setting $F^i\Omega_X^\bullet := \Omega_X^{\geq i}$, and note that the ith graded piece $\mathrm{gr}_i^F \Omega_X^\bullet$

is isomorphic to Ω^i_X, shifted by p. This yields the Hodge-to-de-Rham spectral sequence with first page $E_1^{p,q} = H^q(X, \Omega^p_X) \implies H^{p+q}_{\mathrm{dR}}(X)$. The induced descending filtration $H^n_{\mathrm{dR}}(X) = F^0 \supset F^1 \supset \cdots \supset F^{n+1} = 0$ is the Hodge filtration. One of the culminating results of classical Hodge theory is

Theorem 10. *Let X be a compact Kähler manifold. Then the Hodge-to-de-Rham spectral sequence of X degenerates at E_1.*

A few remarks on this theorem: the degeneracy of the Hodge-to-de-Rham spectral sequence at the E_1 page is equivalent to the fact that

$$F^p H^k(X, \mathbb{C}) / F^{p+1} H^k(X, \mathbb{C}) = H^q(X, \Omega^p_X$$

and also to the fact that

$$\dim H^k(X, \mathbb{C}) = \sum_{p+q=k} (\dim H^q(X, \Omega^p_X).$$

One might hope, then, that degeneracy of this spectral sequence at E_1 for any complex manifold X would imply the existence of a Hodge decomposition as in 1.2 or, even more optimistically, the existence of a Kähler form on X. Unfortunately, neither of these are the case – degeneracy of the Hodge-to-de-Rham spectral sequence at E_1 is a strictly weaker condition than Kähler-ness [21].

It is actually possible to apply Serre's GAGA principle here to give an "algebraic analogue" of the above picture. Rather than sheaves of holomorphic differential forms over a complex manifold X with the usual complex topology, we may consider sheaves of algebraic differential forms over X equipped with the Zariski topology, and the de Rham complex of coherent sheaves (*i.e.* finitely presented $\mathcal{O}_X^{\mathrm{alg}}$-modules in the Zariski topology which are, importantly, locally free) on X. Then Serre's GAGA principle ensures that these two spectral sequences degenerate together. That is, degeneracy at E_1 of the Hodge-to-de-Rham spectral sequence of the de Rham complex holds if and only if degeneration at E_1 of the algebraic Hodge-to-de-Rham spectral sequence holds. As discussed earlier, due to the lack of an algebraic Poincaré lemma, the algebraic de Rham complex is not at all locally exact in the Zariski topology, in contrast to the situation for the ordinary de Rham complex [**?**]. This lack of an algebraic Poincaré lemma was precisely what led us to turn to the p-adic setting for constructing an appropriate comparison isomorphism. We will make the same move here; the existence of a p-adic Poincaré lemma also gives us more freedom when working over p-adic fields.

2.2 The singular case

The usual Hodge-to-de-Rham spectral sequence is useful insofar as it allows us to easily compute the de Rham cohomology, hence the Betti cohomology, of smooth complex varieties. However, these cohomology theories do not necessarily coincide in the case of singular varieties X over arbitrary fields k of characteristic 0, as the space $\Omega^1_{X/k}$ is not locally free near singularities. Hartshorne's *algebraic de Rham cohomology* [11] is the natural analog. Let $Y \to X$ be a closed immersion of an embeddable scheme Y into a smooth scheme X over k. Then the *algebraic de Rham cohomology* of Y is the hypercohomology of the formal completion of $\Omega^\bullet_{X/k}$ along Y. That is,

$$ {}^{\mathrm{alg}}H^q{}_{\mathrm{dR}}(Y) = \mathbb{H}^q(\widehat{X}, \widehat{\Omega}^\bullet_{X/k}). \tag{2.3} $$

Furthermore, this definition is independent of the choice of embedding and the cohomology is a contravariant functor in Y. Illusie [13] showed that this algebraic cohomology theory coincides with the derived de Rham cohomology defined by the cotangent complex for varieties with lci singularities, *i.e.* for which the local ring at every point is a complete intersection ring. More recently, Bhatt [3] generalized this result to *any* finite type morphism of noetherian $\mathbb{Q}$-schemes:

Theorem 11 (Bhatt)**.** *The Hodge-completed derived de Rham cohomology of any finite type morphism of noetherian $\mathbb{Q}$-schemes is canonically isomorphic to Hartshorne's algebraic de Rham cohomology (ignoring filtrations).*

Proof. See [3]. □

Recall that the $\mathbb{Q}$-schemes are those for which the residue field at any point $x \in X$ is characteristic 0, so indeed this is quite a bit more general than Illusie's result. As we shall outline in what follows, the Hodge filtration on the derived de Rham cohomology induces a filtration on the algebraic cohomology, which allow us to define the *derived Hodge-to-de-Rham* spectral sequence to compute algebraic de Rham cohomology for noetherian Q-schemes. This theorem shows that several terms in this spectral sequence do not vanish, and in particular that it does not necessarily degenerate on the E_1 page.

We sketch the construction of the *algebraic de Rham complex*. Following Bhatt, we will provide the affine picture – for more on its globalization to an arbitrary variety over k, see [3]. Let $f : A \to B$ is a finite map of noetherian $\mathbb{Q}$-algebras and $F \to B$ is a presentation of B, where F is a finitely generated polynomial A-algebra (here, a *polynomial A-algebra* is meant to be an A-algebra $A[X_i]_{i \in I}$

in indeterminates $\{X_i\}$). Then the (affine) *algebraic de Rham complex*, written $\Omega^H_{B/A}$, is defined as

$$\Omega^H_{B/A} := \Omega^\bullet_{F/A} \otimes_F \widehat{F}, \tag{2.4}$$

where $\widehat{F}$ is the formal completion of F along $I = \ker(F \to A)$. This definition is independent of the choice of F. In this language, we have the following, more precise version of Theorem 11, also due to Bhatt [3]:

Corollary 3. *Let $f : X \to Y$ be a finite type map of noetherian $\mathbb{Q}$-schemes, and assume that X can be realized as a closed subscheme of a smooth Y-scheme. Then there is a natural filtered $f^{-1}\mathcal{O}_Y$-algebra map*

$$L\widehat{\Omega}^\bullet_{X/Y} \to \Omega^H_{X/Y}$$

that is an equivalence of the underlying algebras.

As a consequence of this theorem, the problem of computing algebraic de Rham cohomology in characteristic 0 is almonst eliminated. In the same paper, Bhatt shows that the algebraic de Rham cohomology can be computed by the completed Amitsur complex for any variety in characteristic 0

The filtration on the target is the *derived Hodge filtration*, induced by the Hodge filtration on the derived de Rham algebra. The derived Hodge filtration yields a spectral sequence interpolating between the derived de Rham cohomology and the algebraic cohomology, with first page

$$E_1^{p,q} : H^q(X, \wedge^p L_{X/k}) \implies H^{p+q}(X, \Omega^H_{X/k}).$$

While this spectral sequence isn't of interest computationally, it is still interesting to compare the derived Hodge filtration with the infinitesimal Hodge filtration or Hodge-Deligne filtrations on derived de Rham cohomology. We have the following proposition, from the same paper.

Proposition 8. *Let X be a finite type k-scheme. There are natural maps*

$$\widehat{\mathrm{dR}}_{X/k} \xrightarrow{a} \Omega^H_{X/k} \xrightarrow{b} \Omega^*_{X/k} \xrightarrow{c} \underline{\Omega}^*_{X/k}$$

of filtered complexes such that $a, c\circ b$, and $c \circ b \circ a$ induce an equivalence on the underlying complexes. In particular, the algebraic de Rham cohomology of X is a summand of the cohomology of $\Omega^*_{X/k}$.

Here $\underline{\Omega}^*_{X/k}$is the Deligne-De Bois Complex, defined in [3]

There are several questions still open in this direction. Of particular interest to me is understanding the precise structure of "cancellation" in the derived Hodge-to-de-Rham spectral sequence. That is, though the cohomology of the derived de Rham complex is unbounded in dimension, Theorem 11 implies that the cohomology of the total complex is finite [3], so there is nontrivial cancellation occurring throughout the spectral sequence. Possible future work includes computing several pages of these sequences explicitly to gain a greater understanding of this cancellation behavior.

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